NONLINEAR FILTRATION OF A LIQUID SUBJECT TO A POWER RESISTANCE LAW

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A method of solving the plane problem for the nonlinear filtration of an incompressible liquid is proposed, assuming a power-type law of resistance, with rectilinear boundaries of the region of motion. The method is demonstrated for the case of filtration through a uniformly-porous wedge.

S. A. Khristianovich [1] first indicated the possibility of using the hodograph of S. A. Chaplygin [2] for studying nonlinear problems in filtration theory. This method was later used with success in [3, 4, 6].

The steady-state motion of an incompressible liquid in a porous medium characterized by a power resistance law is described by the equations:

$$v_x = -\frac{\alpha}{v^n}\frac{\partial P}{\partial x};\ v_y = -\frac{\alpha}{v^n}\frac{\partial P}{\partial y};\ v_x = \frac{\partial \Psi}{\partial y};\ v_y = -\frac{\partial \Psi}{\partial x}. \tag{1}$$

If in system (1) we transform to Chaplygin variables and put

$$\tilde{v} = \exp(\tau/\sqrt{n+1});\ \tilde{P} = Q\exp(n\tau/2\sqrt{n+1}), \tag{2}$$

for the function Q we obtain the Helmholtz equation

$$\frac{\partial^2 Q}{\partial \beta^2} + \frac{\partial^2 Q}{\partial \tau^2} - \frac{n^2}{4(n+1)}Q = 0. \tag{3}$$

Let the liquid filter through a wedge ABCD (Fig. 1) with a base angle β_0. Let us assume that on the face AC $P = P_1 > 0$, on the faces AB and BC $P = P_2 = 0$, and at the point D $v = v_1$ ($\tau = 0$). It is physically obvious that at the vertices A and C $v = \infty$ ($\tau = \infty$). At the point B, as in linear filtration, we may consider that $v = 0$ ($\tau = -\infty$). In the variables τ the region of filtration appears in the form of an infinite strip $2\beta_0$ wide with a slot along the positive semiaxis of τ (Fig. 1). Owing to the presence of the slot (discontinuity), the function Q is not single-sheeted in this region. It is therefore appropriate to limit consideration to the upper part of the strip with $\beta_0 \geq \beta \geq 0$; inside this region Q is single-sheeted.

At the internal points of the region ABD Q is finite. Despite the singularities at points A and B, Q is absolutely integrable with respect to the variable τ. This may be established from a physical consideration

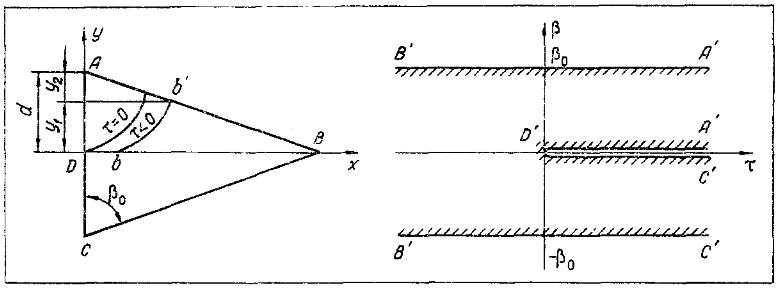

Fig. 1. Region of filtration.

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of the character of the singularities at points A and B. Thus with respect to the function Q a Fourier transformation with respect to the variable τ may be applied in the region of filtration.

For the Fourier transform

$$Q = \int_{-\infty}^{\infty} Q(\xi, \beta) \exp(-i\lambda\xi)\, d\xi$$

we obtain the differential equation

$$\frac{\partial^2 \bar{Q}}{\partial \beta^2} - q^2 \bar{Q} = 0 \quad \left(q^2 = \lambda^2 + \frac{n^2}{4(n+1)}\right)$$

with boundary conditions

$$\bar{Q} = \begin{cases} \bar{Q}(\lambda) = \int_{-\infty}^{\infty} Q(\xi, 0) \exp(-i\lambda\xi)\, d\xi & \text{for } \beta = 0, \\ 0 & \text{for } \beta = \beta_0, \end{cases} \tag{4}$$

the solution of which takes the form

$$\bar{Q} = \bar{Q}(\lambda)(\operatorname{ch} q\beta - \operatorname{cth} q\beta_0 \operatorname{sh} q\beta). \tag{5}$$

The original for Q is determined by the inverse Fourier transformation formula

$$Q = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{Q}(\lambda)(\operatorname{ch} q\beta - \operatorname{cth} q\beta_0 \operatorname{sh} q\beta) \exp i\lambda\tau\, d\lambda. \tag{6}$$

Equation (4) contains the as-yet unknown function $Q(\xi, 0)$. In order to determine this we must satisfy the condition

$$\partial Q / \partial \beta = 0 \ \text{for} \ \beta = 0, \ \tau \leqslant 0. \tag{7}$$

In the plane of the half wedge (Fig. 1) let us consider the line τ = const. For $\tau < 0$ we find y_1 and y_2 by moving successively from point b to point b' with τ = const, and then with $\beta = \beta_0$ along the face of the wedge from b' to A. We shall make use of the Chaplygin equations

$$\begin{aligned} d\tilde{x} &= -\frac{1}{\chi \tilde{v}^{n+1}} \cos\beta\, d\tilde{P} - \frac{1}{\tilde{v}} \sin\beta\, d\tilde{\Psi}; \\ d\tilde{y} &= -\frac{1}{\chi \tilde{v}^{n+1}} \sin\beta\, d\tilde{P} + \frac{1}{\tilde{v}} \cos\beta\, d\tilde{\Psi}. \end{aligned} \tag{8}$$

In determining y_1 we use the second of Eqs. (8), carrying the integration with respect to β from 0 to β_0. In this we employ Eq. (6), changing the order of integration with respect to λ and β. In calculating y_2 we carry out an integration of the second of Eqs. (8) with $\beta = \beta_0$ for τ values between τ and ∞. As before we use (6) with a change in the order of integration with respect to τ and λ. Since according to (7) we should have $y_1 + y_2 = d$, we now find

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{q \operatorname{cth} q\beta_0}{\lambda + i\dfrac{n+2}{2\sqrt{n+1}}} \bar{Q}(\lambda) \exp i\lambda\tau\, d\lambda = \frac{\chi}{\sqrt{n+1}} \exp \frac{n+2}{2\sqrt{n+1}} \tau. \tag{9}$$

The resultant Fourier–Fredholm equation serves to determine the unknown $Q(\xi, 0)$ for $\tau < 0$.

Let us denote the right-hand side of Eq. (9) for any $\tau(-\infty < \tau < \infty)$ and put

$$\psi(\tau) = \begin{cases} \dfrac{\chi}{\sqrt{n+1}} \exp \dfrac{n+2}{2\sqrt{n+1}} \tau & (\tau < 0), \\ 0 & (\tau > 0). \end{cases}$$

We express $\psi(\tau)$ in the form of a sum

$$\psi(\tau) = \psi_+(\tau) + \psi_-(\tau).$$

Here the unknown function

$$\psi_+(\tau) = 0 \ \text{for} \ \tau < 0.$$

We further introduce the notation

$$\overline{Q}(\lambda) = \overline{Q}_{+}(\lambda) + \overline{Q}_{-}(\lambda),$$

where

$$\overline{Q}_{+}(\lambda) = \int_{0}^{\infty} Q_{+}(\xi,\ 0)\exp(-i\lambda\xi)\,d\xi;\ \overline{Q}_{-}(\lambda) = \int_{-\infty}^{0} Q_{-}(\xi,\ 0)\exp(-i\lambda\xi)\,d\xi;$$

$$\overline{\psi}_{+}(\lambda) = \int_{0}^{\infty} \psi_{+}(\xi)\exp(-i\lambda\xi)\,d\xi;\ \overline{\psi}_{-}(\lambda) = \int_{-\infty}^{0} \psi_{-}(\xi)\exp(-i\lambda\xi)\,d\xi.$$

It is obvious that

$$\overline{\psi}_{-}(\lambda) = \frac{i\chi}{\sqrt{n+1}\,[\lambda + i(n+2)/2\sqrt{n+1}]};\ \overline{Q}_{+}(\lambda) = -\frac{i}{\lambda - in/2\sqrt{n+1}}.$$

If we put

$$K(\lambda) = q\,\mathrm{cth}\,q\beta_0/\left(\lambda + i\frac{n+2}{2\sqrt{n+1}}\right),$$

Eq. (9) may be written in the form

$$K(\lambda)\overline{Q}_{-}(\lambda) - i\frac{K(\lambda)}{\lambda - in/2\sqrt{n+1}} = \frac{\chi}{\sqrt{n+1}\left(\lambda + i\frac{n+2}{2\sqrt{n+1}}\right)} - i\overline{\psi}_{+}(\lambda). \qquad (10)$$

We study Eq. (10) by the Wiener–Hopf method [5]. First let us factorize the meromorphic function $K(\lambda)$. Putting

$$K(\lambda) = \tilde{\varphi}_{+}(\lambda)\tilde{\varphi}_{-}(\lambda),$$

where $\tilde{\varphi}_{+}(\lambda)$ and $\tilde{\varphi}_{-}(\lambda)$ are whole functions not having zeros in the lower half plane (including the real axis) and the upper half plane (also embracing the real axis) respectively, we easily find that

$$\tilde{\varphi}_{+}(\lambda) = \frac{\prod_{k=1}^{\infty}\frac{2}{2k-1}(\lambda - is_k)\exp\frac{\lambda}{is_k}}{\prod_{k=1}^{\infty}\frac{1}{k}(\lambda - ir_k)\exp\frac{\lambda}{ir_k}};$$

$$\tilde{\varphi}_{-}(\lambda) = \frac{\prod_{k=1}^{\infty}\frac{2}{2k-1}(\lambda + is_k)\exp\left(-\frac{\lambda}{is_k}\right)}{\prod_{k=1}^{\infty}\frac{1}{k}(\lambda + ir_k)\exp\left(-\frac{\lambda}{ir_k}\right)} \cdot \frac{1}{\beta_0\left(\lambda + i\frac{n+2}{2\sqrt{n+1}}\right)}.$$

Here

$$r_k = \sqrt{\frac{k^2\pi^2}{\beta_0^2} + \frac{n^2}{4(n+1)}};\ s_k = \sqrt{\frac{(2k-1)^2\pi^2}{4\beta_0^2} + \frac{n^2}{4(n+1)}}.$$

Since the series $\sum_{k=1}^{\infty} r_k - s_k/r_k s_k$ converges, another factorization is permissible:

$$K(\lambda) = \varphi_{-}(\lambda)\varphi_{+}(\lambda)\frac{1}{\lambda + i\frac{n+2}{2\sqrt{n+1}}},$$

where

$$\varphi_{-}(\lambda) = \prod_{k=1}^{\infty}\frac{2k}{2k-1}\,\frac{\lambda + is_k}{\lambda + ir_k};\ \beta_0\varphi_{+}(\lambda) = \prod_{k=1}^{\infty}\frac{2k}{2k-1}\,\frac{\lambda - is_k}{\lambda - ir_k}.$$

Equation (10) may now be written in the form

$$\left[\overline{Q}_{-}(\lambda) - \frac{i}{\lambda - in/2\sqrt{n+1}}\right]\varphi_{-}(\lambda) = \frac{\chi}{\varphi_{+}(\lambda)\sqrt{n+1}} - \frac{i\overline{\psi}_{+}(\lambda)}{\varphi_{+}(\lambda)}\left(\lambda + i\frac{n+2}{2\sqrt{n+1}}\right). \qquad (11)$$

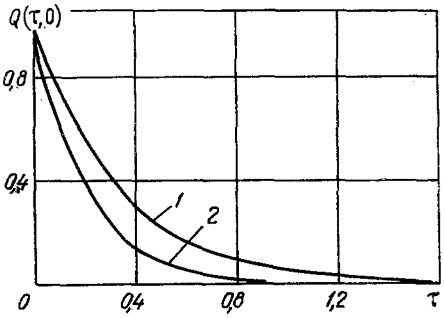

Fig. 2. Curves of the function $Q(\tau, 0)$ $(\tau < 0)$: 1) $\beta_0 = \pi/4$; 2) $\beta_0 = \pi/6$.

The left-hand side of Eq. (11) has a pole at the point $\lambda = in/2\sqrt{(n+1)}$ in the upper plane with a certain multiplicity m. If we multiply (11) by $(\lambda - in/2\sqrt{n+1})^m$ we obtain functions on the left and right which are regular in the upper and lower half plane respectively and also coincident and regular in a certain strip including the real axis. Thus, starting from the principle of analytical continuation, we find that the result of the multiplication of (11) by $(\lambda - in/2\sqrt{n+1})^m$ constitutes a whole function over all the λ plane; since in the half planes of regularity $\overline{Q}_-(\lambda) = O(1/\lambda)$; $\overline{\psi}_+(\lambda) = O(1/\lambda)$; $\varphi_+(\lambda) = O(\sqrt{\lambda})$; $\varphi_-(\lambda) = O(\sqrt{\lambda})$, this function may be expressed by a polynomial of degree m−1 containing m parameters. Since $Q_-(\tau, 0)$ should satisfy two conditions, namely 1) the residue of $\overline{Q}_-(\lambda)$ at the point $\lambda = in/2\sqrt{n+1}$ should be zero, otherwise as $\tau \to -\infty$ $Q_-(\tau, 0)$ would be of the order of $\exp(n\tau/2\sqrt{n+1}) \to \infty$; and 2) $Q(0, 0) = 1$, we find that the polynomial should contain altogether two parameters. This means a polynomial of the first degree.

Thus we have:

$$\overline{Q}_-(\lambda) = \frac{i}{\lambda - in/2\sqrt{n+1}} + \frac{a+ib}{(\lambda - in/2\sqrt{n+1})^2\varphi_-(\lambda)}. \tag{12}$$

In view of the realness of $Q_-(\tau, 0)$, a and b are here real parameters. Allowing for condition 1) we obtain from (12):

$$i - a\frac{\varphi'_-(in/2\sqrt{n+1})}{\varphi^2_-(in/2\sqrt{n+1})} + \frac{bn}{2\sqrt{n+1}}\frac{\varphi'_-(in/2\sqrt{n+1})}{\varphi^2_-(in/2\sqrt{n+1})} + \frac{ib}{\varphi_-(in/2\sqrt{n+1})} = 0.$$

If we introduce a function of the real variable $z \geq 0$

$$\Phi(z, n, \beta) = \prod_{k=1}^{\infty}\frac{2k-1}{2k}\frac{z+r_k}{z+s_k},$$

instead of the preceding equation we may write

$$-a\Phi'(n/2\sqrt{n+1}) + b\left[\frac{n}{2\sqrt{n+1}}\Phi'(n/2\sqrt{n+1}) - \Phi(n/2\sqrt{n+1})\right] = 1. \tag{13}$$

Since in the lower half plane $\overline{Q}_-(\lambda) = O(1/\lambda)$, the original for the function $\overline{Q}_-(\lambda)$ may be obtained by using the residue theorem. Since $\overline{Q}_-(\lambda)$ has simple poles at the points $\lambda = is_k$, on considering condition 1) we have

$$Q_-(\tau, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{Q}_-(\lambda)\exp i\lambda\tau\, d\lambda = \frac{a}{\beta_0^2}\sum_{k=1}^{\infty}\frac{\exp s_k\tau}{(s_k + n/2\sqrt{n+1})^2 s_k\Phi(s_k)}$$

$$-\frac{b}{\beta_0^2}\sum_{k=1}^{\infty}\frac{\exp s_k\tau}{(s_k + n/2\sqrt{n+1})^2\Phi(s_k)}. \tag{14}$$

Using condition 2) we obtain

$$\beta_0^2 = a\sum_{k=1}^{\infty}\frac{1}{(s_k + n/2\sqrt{n+1})^2 s_k\Phi(s_k)} - b\sum_{k=1}^{\infty}\frac{1}{(s_k + n/2\sqrt{n+1})^2\Phi(s_k)}. \tag{15}$$

The system of equations (13), (15) determines the values of a and b:

$$a = -\frac{B - \beta_0^2\left[\Phi(n/2\sqrt{n+1}) - \frac{n}{2\sqrt{n+1}}\Phi'(n/2\sqrt{n+1})\right]}{A\left[\Phi(n/2\sqrt{n+1}) - \frac{n}{2\sqrt{n+1}}\Phi'(n/2\sqrt{n+1})\right] + B\Phi'(n/2\sqrt{n+1})};$$

$$b = -\frac{A + \beta_0^2\Phi'(n/2\sqrt{n+1})}{A\left[\Phi(n/2\sqrt{n+1}) - \frac{n}{2\sqrt{n+1}}\Phi'(n/2\sqrt{n+1})\right] + B\Phi'(n/2\sqrt{n+1})}, \tag{16}$$

where

$$A=\sum_{k=1}^{\infty}\frac{1}{(s_k+n/2\sqrt{n+1})^2 s_k\,\Phi(s_k)};\quad B=\sum_{k=1}^{\infty}\frac{1}{(s_k+n/2\sqrt{n+1})^2\,\Phi(s_k)}.$$

In order to determine the filtration parameter χ, we use the expression for $d\tilde{x}$ on the symmetry axis:

$$\chi\, d\tilde{x}=-\frac{n}{2\sqrt{n+1}}Q_-(\tau,\ 0)\exp\left(-\frac{n+2}{2\sqrt{n+1}}\tau\right)d\tau-\frac{\partial[Q_-(\tau,\ 0)]}{\partial\tau}\exp\left(-\frac{n+2}{2\sqrt{n+1}}\tau\right). \qquad (17)$$

Integrating Eq. (17) with respect to τ from 0 to $-\infty$ and using (14) we obtain

$$\chi=\frac{\operatorname{ctg}\beta_0}{\beta_0^2}\sum_{k=1}^{\infty}\frac{a-bs_k}{(s_k-n/2\sqrt{n+1})\left(s_k-\frac{n+2}{2\sqrt{n+1}}\right)s_k\Phi(s_k)}. \qquad (18)$$

Figure 2 shows the curves of $Q(\tau,\ 0)$ for n = 1 and $\beta_0=\pi/4$; $\beta_0=\pi/6$. In these cases the parameter χ respectively equalled 1.99 and 3.55.

NOTATION

v	is the filtration velocity;
$v_x = v\cos\beta$, $v_y = v\sin\beta$	its projections on the coordinate axes;
β	is the angle between the filtration velocity vector and the Dx axis;
v_1	is the characteristic velocity;
$\tilde{v} = v/v_1$, $\tilde{v}_x = v_x/v_1$, $\tilde{v}_y = v_y/v_1$	are the dimensionless velocity and its projections on the coordinate axes;
P	is the pressure;
P_1	is the characteristic pressure;
$\tilde{P} = P/P_1$	is the dimensionless pressure;
d	is the characteristic dimension;
$\tilde{x} = x/d$, $\tilde{y} = y/d$	are the dimensionless coordinates;
$n \geq 0$	is the degree of filtration; at n = 0 the filtration obeys the D'Arcy law, at n > 0 the filtration is nonlinear;
α	is the constant of power-law filtration depending on the physical properties of the porous material;
$\chi = v_1^{n+1}d/P_1\alpha$	is the dimensionless filtration parameter;
Ψ	is the current function;
λ	is the Fourier parameter.

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